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DISCUSSION
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Sequential dominance and weighted utilitarianism

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Abstract

Ok and Lambert (1999) show that one does not have to be a utilitarian to accept Atkinson and Bourguignon's (1987) sequential generalized Lorenz dominance criterion, because the latter is also supported by a much wider class of aggregation functions. We take a minimal stance: we show that it suffices to be a weighted utilitarian —with higher weights for the more needy— to accept it. We also discuss some possible extensions.

1 Motivation

When all households are homogeneous in all relevant non-income characteristics —e.g., households size, average age or handicap level of the household members, and so on— there are many ways to assess the corresponding income distributions in terms of poverty, inequality or welfare (see, e.g., Lambert, 2001 for an overview). In particular, the generalized Lorenz dominance (GLD) criterion (Shorrocks, 1983) is widely accepted and used in economics. Unfortunately, such tools are not well-suited to make reasonable comparisons in practice, because "*At the heart of any distributional analysis, there is the problem of allowing for differences in people's non-income characteristics*" (Cowell and Mercader-Prats, 1999). The standard way to proceed is (i) to convert the household income distribution of heterogeneous households into an 'equivalent income' distribution of reference types and (ii) to apply a social evaluation tool —such as the GLD criterion— to the equivalent income distribution.[1]

Almost twenty years ago, Atkinson and Bourguignon (1987) proposed a robust approach, which is not based on the use of a specific equivalence scale to cardinalize needs, but on an

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[1] There is a debate whether one should weight the equivalent incomes by the equivalence scales or by the household size; see Ebert (1997), Ebert and Moyes (2003), Shorrocks (2004) and Capéau and Ooghe (2004) for a discussion. We will sidestep this issue here.

ordinal classification of all households into different need groups. To apply their sequential generalized Lorenz dominance (SGLD) criterion, one has to check —on the basis of the GLD criterion— whether (i) the most needy households of one distribution, say distribution A, dominate the most needy of another distribution, say B, (ii) whether the most and second most needy households of distribution A together also dominate the most and second most needy of distribution B, and so on. Atkinson and Bourguignon (1987) also show that the the SGLD criterion has a strong normative support, for which we may regard the calculation of social welfare to be split up in three succesive steps: the calculation of (1) the welfare of each household (depending on household income and needs type) (2) the welfare of each subgroup of households with the same needs and (3) the total welfare. In essence, Atkinson and Bourguignon (1987) show that the SGLD criterion is equivalent with unanimity among social welfare functions using a wide class of utility profiles in step (1) and a utilitarian approach in steps (2) and (3).

Some authors have put considerable effort into extending the SGLD criterion, e.g., to deal with a wider class of utility profiles, poverty, changing demographics and/or the principle of diminishing transfers (see: Bourguignon, 1989; Atkinson, 1992; Jenkins and Lambert, 1993; Chambaz and Maurin, 1998; Moyes, 1999; Lambert and Ramos, 2002). Others have contributed to the understanding of the SGLD criterion by providing different characterizations (see: Ok and Lambert, 1999; Ebert, 2000; Ooghe and Lambert, 2005). In particular, Ok and Lambert (1999) show that one does not have to be a utilitarian in step (3) to accept the SGLD criterion. They provide the 'largest' class of social welfare functions in step (3) which supports the SGLD criterion, assuming Atkinson and Bourguignon's class of utility profiles in step (1) and utilitarianism in step (2). We present both Atkinson and Bourguignon's (1987) classical as well as Ok and Lambert's (1999) result in section 2.

In this note, we take an opposite view. In section 3, we present the 'smallest' class of household utility profiles which can be used in step (1) to support the SGLD criterion, assuming utilitarianism in steps (2) and (3). In particular, this class turns out to contain only profiles in which the welfare of a household is multiplicatively separable in utility of income and needs. As a result, the SGLD criterion is equivalent to unanimity among weighted utilitarian social welfare functions, with higher weights assigned to higher needs groups. In a final section 4, we discuss some possible extensions.

2 Characterizations of the SGLD criterion

We introduce some notation to define the SGLD criterion. Consider household incomes $y \in \mathbb{R}_+$ and household types $k \in \mathbb{K} = \{1, ..., K\}$, ordered from least (1) to most needy (K), given the same household income. For example, k could be household size. A heterogeneous distribution consists of (i) proportions of type k households, denoted p_k, with $\sum_{k\in\mathbb{K}} p_k = 1$, and (ii) income distribution functions of type k households, denoted F_k, defined over a common finite support $[0, z]$. For the moment we assume p_k to be the same for all distributions (no changes in demographics); we abbreviate a distribution as a list $\mathbf{F} = (F_1, \ldots, F_K)$. We are ready to define the sequential generalized Lorenz dominance criterion.

DEFINITION: *A distribution* $\mathbf{F}$ *dominates a distribution* $\mathbf{G}$ *according to the sequential generalized Lorenz dominance (SGLD) criterion, denoted* $\mathbf{F} \succsim_{SGLD} \mathbf{G}$*, if and only if for all* $y \in [0, z]$ *and for all* $k \in \mathbb{K}$ *we have* $\sum_{i=k}^{K} p_i \int_0^y [F_i(x) - G_i(x)]\, dx \leq 0$.

We briefly present the classical characterization of the SGLD criterion by Atkinson and Bourguignon (1987) as well as the 'maximal' characterization of Ok and Lambert (1999).[2] We give an alternative 'minimal' characterization in the next section.

Atkinson and Bourguignon (1987) consider utilitarian social welfare functions, i.e., the social welfare of a distribution $\mathbf{F}$ is defined as the average utility of income, or

$$W(\mathbf{F}; \mathbf{U}) = \sum_{k\in\mathbb{K}} p_k \int_0^z U_k(y)\, dF_k(y), \tag{1}$$

with $U_k : \mathbb{R}_+ \to \mathbb{R}$ (twice differentiable) household utility functions, one for each type $k \in \mathbb{K}$. They consider utility profiles $\mathbf{U} = (U_1, U_2, \ldots, U_K)$ satisfying the following four properties: (AB1) the marginal utility U_k' is positive, i.e. $U_k' \geq 0$, for all types $k \in \mathbb{K}$, (AB2) the marginal utility is decreasing, i.e., $U_k'' \leq 0$, for all types $k \in \mathbb{K}$, (AB3) a household of needs type k has a higher marginal utility compared to a household with needs type $k-1$ (for the same household income), i.e., $U_k' \geq U_{k-1}'$, for all $k = 2, \ldots, K$, and (AB4) the difference in marginal utility defined in (AB3) decreases with income, i.e., $U_k'' \leq U_{k-1}''$, for all $k = 2, \ldots, K$. Call $\mathbb{U}_{AB}$ the family of utility profiles $\mathbf{U}$ satisfying properties AB1-AB4. We get:

ATKINSON & BOURGUIGNON (1987): $\mathbf{F} \succsim_{SGLD} \mathbf{G}$ *if and only if* $W(\mathbf{F}; \mathbf{U}) \geq W(\mathbf{G}; \mathbf{U})$ *for all profiles* $\mathbf{U}$ *in* $\mathbb{U}_{AB}$.

Ok and Lambert (1999) show that one does not have to be a utilitarian to accept the SGLD criterion. First, they measure household well-being via profiles $\mathbf{U}$ in $\mathbb{U}_{AB}$. Second, they assume that the welfare level of a group of households with the same needs can be measured by their average utility of income expressed per capita of the whole population, more precisely, $W_k(\mathbf{F}; \mathbf{U}) = p_k \int_0^z U_k(y)\, dF_k(y)$. Third, in contrast with Atkinson and Bourguignon (1987),

[2] We refer the reader to the original papers for more detailed discussion and to Ebert (2000) and Ooghe and Lambert (2005) for alternative but, in the current context, less relevant characterizations.

they consider a needs-based aggregation function to aggregate the subgroup welfare levels. More precisely, an aggregation function $V : \mathbb{R}^K \to \mathbb{R}$ is needs-based if and only if

$$V(a_1, \ldots, a_{k-1}, a_k, \ldots, a_K) \geq V(a_1, \ldots, a_{k-1} + \epsilon, a_k - \epsilon, \ldots, a_K), \forall \epsilon \geq 0, \forall k = 2, \ldots, K.$$

Call $\mathbb{V}_{OL}$ the family of needs-based aggregation functions V. We get:

Ok & Lambert (1999): $\mathbf{F} \succsim_{SGLD} \mathbf{G}$ *if and only if* $V(W_1(\mathbf{F};\mathbf{U}), \ldots, W_K(\mathbf{F};\mathbf{U})) \geq V(W_1(\mathbf{G};\mathbf{U}), \ldots, W_K(\mathbf{G};\mathbf{U}))$ *for all needs-based aggregation functions* V *in* $\mathbb{V}_{OL}$ *and all profiles* $\mathbf{U}$ *in* $\mathbb{U}_{AB}$.

The characterization by Ok and Lambert (1999) is 'maximal' in the sense that they look for the widest class of aggregation functions (for subgroup welfare levels) which, ceteris paribus, supports the SGLD criterion. Our alternative characterization can be considered as 'minimal' in an analogous way: we look for the smallest class of household utility functions, ceteris paribus, which can do the job.

3 A new characterization

Denote the class of (twice differentiable) increasing and concave utility functions by $\mathcal{U} = \{U : \mathbb{R}_+ \to \mathbb{R} \,|U' \geq 0 \text{ and } U'' \leq 0\}$ and the class of positive and increasing weight vectors by $\mathcal{W} = \left\{\mathbf{w} \in \mathbb{R}_+^K \,|w_1 \leq w_2 \leq \ldots \leq w_K\right\}$. We define $\mathbb{U}$ as the family of utility profiles $\mathbf{U} = (U_1, \ldots, U_K) \equiv (w_1 U, \ldots, w_K U)$ with $U \in \mathcal{U}$ and $\mathbf{w} \in \mathcal{W}$. It is easy to verify that $\mathbb{U} \subset \mathbb{U}_{AB}$. Still, we get:

Proposition 1. $\mathbf{F} \succsim_{SGLD} \mathbf{G}$ *if and only if* $W(\mathbf{F};\mathbf{U}) \geq W(\mathbf{G};\mathbf{U})$ *for all profiles* $\mathbf{U}$ *in* $\mathbb{U}$.

Proof. Because $\mathbb{U} \subset \mathbb{U}_{AB}$, the direction $\Rightarrow$ follows from Atkinson and Bourguignon (1987). We prove the other direction. Consider two arbitrary distributions $\mathbf{F}$ and $\mathbf{G}$. For ease of exposition, define, for each type k, the functions

$$H_k^1 : [0, z] \to \mathbb{R} : y \mapsto F_k(y) - G_k(y) \text{ and } H_k^2 : [0, z] \to \mathbb{R} : y \mapsto \int_0^y H_k^1(x)\, dx.$$

The welfare dominance statement $W(\mathbf{F};\mathbf{U}) \geq W(\mathbf{G};\mathbf{U})$ for all profiles $\mathbf{U}$ in $\mathbb{U}$ is equivalent with

$$\int_0^z U(y)\, d\left\{\sum_{k\in\mathbb{K}} p_k w_k H_k^1(y)\right\} \geq 0 \text{ for all } U \in \mathcal{U} \text{ and for all } \mathbf{w} \in \mathcal{W}. \quad (2)$$

Using Lambert (2001, lemma 3.1, p.54), the first part of equation (2), more precisely

$$\int_0^z U(y)\, d\left\{\sum_{k\in\mathbb{K}} p_k w_k H_k^1(y)\right\} \geq 0 \text{ for all } U \in \mathcal{U}$$

is equivalent with

$$\int_0^y \left\{ \sum_{k\in\mathbb{K}} p_k w_k H_k^1(x) \right\} dx \leq 0 \text{ for all } y \in [0,z].$$

Using the definition of the functions H_k^2, equation (2) is equivalent with

$$\sum_{k\in\mathbb{K}} p_k w_k H_k^2(y) \leq 0 \text{ for all } y \in [0,z] \text{ and for all } \mathbf{w} \in \mathcal{W}. \tag{3}$$

Defining $a_1 = w_1 \geq 0$ and $a_k = w_k - w_{k-1} \geq 0$, for all $k = 2, \ldots, K$, we can equivalently rewrite equation (3) as

$$\sum_{k\in\mathbb{K}} a_k \left\{ \sum_{i=k}^{K} p_i H_i^2(y) \right\} \leq 0 \text{ for all } y \in [0,z] \text{ and for all } (a_1, a_2, \ldots, a_K) \in \mathbb{R}_+^K. \tag{4}$$

Using Atkinson and Bourguignon (1987, lemma 2, p.368), equation (4) holds if and only if $\sum_{i=k}^{K} p_i H_i^2(y) \leq 0$ for all $y \in [0,z]$ and for all $k \in \mathbb{K}$, which is (by definition) the SGLD criterion. □

Proposition 1 tells us that it suffices to look at household utility functions which are multiplicatively separable in utility of needs (as measured by the weight vector $\mathbf{w}$) and utility of income (as measured by the utility function U). It allows us to reinterpret the SGLD criterion as equivalent to weighted utilitarianism, where more needy households receive larger weights. The next section concludes, discussing some possible extensions.

4 Some extensions

First, in the spirit of Fleurbaey, Hagneré and Trannoy (2003), we could put additional lower and upper bounds on the weights to obtain more complete quasi-orderings. We define a set of bounded weights as $\mathcal{W}(\underline{\mathbf{w}}, \overline{\mathbf{w}}) = \{\mathbf{w} \in \mathcal{W} \,|\underline{w}_i \leq w_i \leq \overline{w}_i \text{ for all } i \in \mathbb{K}\}$ for lower and upper bound vectors $\underline{\mathbf{w}} = (\underline{w}_1, \ldots, \underline{w}_K) \leq \overline{\mathbf{w}} = (\overline{w}_1, \ldots, \overline{w}_K)$ and let $\mathbb{U}(\underline{\mathbf{w}}, \overline{\mathbf{w}})$ be the family of utility profiles $\mathbf{U} \equiv (w_1 U, \ldots, w_K U)$ with $U \in \mathcal{U}$ and $\mathbf{w} \in \mathcal{W}(\underline{\mathbf{w}}, \overline{\mathbf{w}})$. It is easy to verify that $\mathbb{U}(\underline{\mathbf{w}}, \overline{\mathbf{w}}) \subset \mathbb{U}$. We get:

Proposition 2. *$W(\mathbf{F}; \mathbf{U}) \geq W(\mathbf{G}; \mathbf{U})$ for all profiles $\mathbf{U}$ in $\mathbb{U}(\underline{\mathbf{w}}, \overline{\mathbf{w}})$ if and only if*

$$\sum_{k\in\mathbb{K}} p_k w_k H_k^2(y) \leq 0 \textit{ for all } y \in [0,z] \textit{ and for all } \mathbf{w} \in \mathcal{W}(\underline{\mathbf{w}}, \overline{\mathbf{w}}).$$

Proof. Follows directly from the proof of proposition 1 up to equation (3). □

As $\sum_{k\in\mathbb{K}} p_k w_k H_k^2(y)$ is a continuous function and $\mathcal{W}(\underline{\mathbf{w}}, \overline{\mathbf{w}})$ is a compact set, the criterion in proposition 2 can be implemented, e.g., by checking whether $\max_{\mathbf{w}\in\mathcal{W}(\underline{\mathbf{w}},\overline{\mathbf{w}})} \sum_{k\in\mathbb{K}} p_k w_k H_k^2(y) \leq 0$ holds for all $y \in [0, z]$.

Second, one could also use a different strategy to restrict the set of weights. The fact that $w_i \geq w_j$ for all $i \geq j$ in the set $\mathcal{W}$ guarantees that a (marginal) money transfer from a poor and needy household (say type i) to a richer and less needy household (with type $j \leq i$), cannot increase social welfare. One could additionally impose that (for the same household incomes) such regressive transfers have a stronger (negative) effect on social welfare when both households become more needy (changing needs types from i and j to $i+1$ and $j+1$), ceteris paribus. To obtain this, we could focus on a set[3]

$$\mathcal{W}^{\circ} = \{\mathbf{w} \in \mathcal{W} \,|\, w_k - w_{k-1} \geq w_{k-1} - w_{k-2} \text{ for all } k = 3, \ldots, K \text{ and } w_2 - w_1 \geq w_1\}$$

and let $\mathbb{U}^{\circ} \subset \mathbb{U}$ denote the corresponding family of utility profiles. This leads to a weighted version of the sequential generalized Lorenz dominance criterion, with higher weights for the more needy.

Proposition 3. *$W(\mathbf{F}; \mathbf{U}) \geq W(\mathbf{G}; \mathbf{U})$ for all profiles $\mathbf{U}$ in $\mathbb{U}^{\circ}$ if and only if*

$$\sum_{i=k}^{K} (k - i + 1)\, p_i H_i^2(y) \leq 0 \text{ for all } y \in [0, z] \text{ and for all } k \in \mathbb{K}.$$

Proof. Reconsider the proof of proposition 1 up to equation (4). Defining $b_1 = a_1 \geq 0$ and $b_k = a_k - a_{k-1} \geq 0$, for all $k = 2, \ldots, K$, we can equivalently rewrite equation (4) as

$$\sum_{k\in\mathbb{K}} b_k \left\{ \sum_{i=k}^{K} (i - k + 1)\, p_i H_i^2(y) \right\} \leq 0 \text{ for all } y \in [0, z] \text{ and for all } (b_1, b_2, \ldots, b_K) \in \mathbb{R}_+^K. \quad (5)$$

As before, equation (5) holds if and only if $\sum_{i=k}^{K} (i - k + 1)\, p_i H_i^2(y) \leq 0$ for all $y \in [0, z]$ and for all $k \in \mathbb{K}$. □

Notice that proposition 3 can be extended in a straightforward way, by restricting the signs of the differences $w_k - w_{k-1} - (w_{k-1} - w_{k-2})$, and so on. The corresponding sequential generalized Lorenz dominance criteria will put increasingly more weight on the higher needs classes. In the limit, one obtains a Rawlsian criterion: a distribution $\mathbf{F}$ is better than $\mathbf{G}$ if and only if F_K generalized Lorenz dominates G_K.

[3] Notice that the last condition $w_2 - w_1 \geq w_1$ (possibly in combination with previous ones) implies that a (marginal) money transfer from a poor and needy household (of type i) to a richer (say income y) and less needy household (of type $i-1$) is worse for social welfare than simply taking the same marginal amount of money away from a household of type 1 and income y.

Finally, if demographics are allowed to change, we must compare (extended) distributions denoted by $\overline{\mathbf{F}} = (\mathbf{p}; \mathbf{F})$, which consists of both proportions $\mathbf{p} = (p_1, ..., p_K)$ and distribution functions $\mathbf{F} = (F_1, \ldots, F_K)$; an alternative (extended) distribution will be denoted by $\overline{\mathbf{G}} = (\mathbf{q}; \mathbf{G})$. To extend the above proposition to deal with changing demographics one has to impose an additional level condition on the common utility function U. Either one could assume $U(z) = 0$, which implies $U_i(z) = U_j(z)$ for all $i, j \in \mathbb{K}$, for each profile $\mathbf{U}$ in $\mathbb{U}$.[4] Welfare dominance for this restricted class of utility profiles is equivalent with Jenkins and Lambert's (1993) and Chambaz and Maurin's (1998) extension of the SGLD criterion:

$$\sum_{i=k}^{K} \int_0^y \left[p_i F_i(x) - q_i G_i(x)\right] dx \leq 0 \text{ for all } y \in [0, z] \text{ and for all } k \in \mathbb{K}. \tag{6}$$

Or one could impose the weaker condition $U \leq 0$ on $[0, z]$ (and thus $U_{k-1} \geq U_k$ on $[0, z]$ for all $k = 2, \ldots, K$ for each profile $\mathbf{U}$ in $\mathbb{U}$) to get Moyes' (1999) extension, which equals condition (6) together with the conditions $\sum_{i=k}^{K} (p_k - q_k) \leq 0$ for all $k \in \mathbb{K}$.

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